A Global Characterization of Envy-free Truthful Scheduling of Two Tasks

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Abstract

We study envy-free and truthful mechanisms for domains with additive valuations, like the ones that arise in scheduling on unrelated machines. We investigate the allocation functions that are both weakly monotone (truthful) and locally efficient (envy-free), in the case of only two tasks, but *many* players. We show that the only allocation functions that satisfy both conditions are affine minimizers, with strong restrictions on the parameters of the affine minimizer. As a further result, we provide a common payment function, i.e., a single mechanism that is both truthful and envy-free.

For additive combinatorial auctions our approach leads us (only) to a non- affine maximizer similar to the counterexample of Lavi et al. [27]. Thus our result demonstrates the inherent difference between the scheduling and the auctions domain, and inspires new questions related to the classic problem of characterizing truthfulness in additive domains.

Since for two tasks, the so called *anonymous* allocations turn out to be envy-free, we obtain a characterization of anonymous (two-task) mechanisms in addition.

1 Introduction

We are interested in characterizing the class of *deterministic* mechanisms that are both *incentive-compatible* and *envy-free* for domains with *additive* valuations[1]. Such valuations arise naturally in many interesting problems, like for instance scheduling on unrelated machines, and combinatorial auctions with additive bidders. We describe the whole setting as a scheduling problem. There are n machines (agents) and m tasks, and the processing time needed for a task j to run on machine i is t_{ij}, and is *privately* known only to the agent that owns the machine. Incentive-compatibility assures that no player can gain by misreporting her true values, while envy-freeness that no individual is envious of the combination of tasks and payments given to other players.

Incentive-compatibility. The scheduling setting was originally proposed by Nisan and Ronen, in their seminal paper [33] that pioneered the field of Algorithmic Mechanism Design, as a vehicle to explore the potentiality/limitations of truthful mechanisms in optimization problems. It was demonstrated that not all objectives can be truthfully optimized, *even by non polynomial-time algorithms.* In particular, a standard performance criterion in the scheduling literature is makespan minimization (i.e. minimizing the maximum completion time of a machine), which is radically different than the common, well-studied social welfare maximization objective in economics. Nisan and Ronen showed that it is impossible to design deterministic truthful mechanisms with approximation guarantee better than 2, and they conjectured that VCG [37, 13, 21] (that achieves the rather unattractive ratio of n) is optimal among truthful mechanisms. The conjecture still remains open; the constant lower bound has been slightly improved to 2.41 for three machines [9], and later to 2.61 for n machines [26].

One of the reasons that make the problem particularly difficult, is the lack of a useful characterization of the allocation functions used by incentive-compatible mechanisms for restricted domains. There are two types of characterizations that dominate the literature of Mechanism Design. Characterizations of the first type, like Weak Monotonicity [31, 27, 36] or Cycle Monotonicity [35], describe the implementable allocations in a *local* fashion. Roughly, these are properties that describe the restrictions imposed on a *single* player's possible allocations with respect to his declarations. The second type characterizes the implementable allocations in a more *global* fashion. The most important result of this type is due to Roberts who showed that for *unrestricted* domains (where all possible valuations over outcomes are

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[1] A preliminary version of this work has appeared in [12]

allowed) the only implementable social choice rules are a simple generalization of VCG mechanisms, namely *affine maximizers* [25].

Since we follow the scheduling notation, and the players are cost minimizers instead of utility maximizers, it will be useful to define affine minimizers.

Definition 1. [Affine Minimizers] We say that an allocation function is an *affine minimizer* if there exist nonnegative constants λ_i, one for each player $i = 1, \ldots, n$, and γ_a one for each allocation a, such that the mechanism selects the allocation a, that minimizes

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i \cdot a_{ij} \cdot t_{ij} + \gamma_a,$$

where a_{ij} is 1 if player i gets task j according to a, and 0 otherwise.

Both characterizations of the first type have been proved very useful in the design of truthful mechanisms, but only in domains that are very restricted; there exist *non-VCG* deterministic monotone algorithms with optimal performance for single-parameter[2] valuation domains (e.g. scheduling on related machines [1, 11], single-minded combinatorial auctions [29, 7]), and cycle-monotone algorithms for multi-dimensional domains with only two possible values [28]. However, for more general multi-dimensional domains such characterizations did not seem to be informative so far. A global, Roberts-like characterization would be much more useful. Unfortunately, Roberts' requirement of unrestricted valuations does not apply to many realistic setups with richer structure, like combinatorial auctions and scheduling where inherently, externalities[3] make not much sense. Only for the special cases of two players, when *all items/tasks must be allocated*[4], is a global characterization known [19, 10].

Since the problem has remained open for so long, there have been efforts to impose extra conditions on top of incentive-compatibility, in order to restrict further the class of possible mechanisms, and try to make the problem easier to attack. Lavi et al. [27] showed that assuming a restriction analogous to the Arrowian Independence of Irrelevant Alternatives, the only truthful mechanisms (in order based domains) are so-called "almost-" affine maximizers.

Anonymous mechanisms. A natural restriction for truthful additive mechanisms is *anonymity*, i.e. the allocation should not depend on the identities of the players. We use the same (rather weak) definition of anonymity as [2]:

Definition 2. An allocation rule is called *anonymous*, if for any input matrix $t = [t_{ij}]_{n\times m}$ with pairwise different bids (rows), exchanging the inputs t_i and t_j of arbitrary two players (leaving the rest unchanged), results in exchanging the allocations a_i and a_j of these players.

Ashlagi et al. [2] succeeded in proving the Nisan-Ronen conjecture for anonymous scheduling mechanisms: no anonymous truthful mechanism can achieve a better than $\min(n, m)$ approximation of the optimum makespan. Their lower-bound proof did not need a characterization of anonymous truthful mechanisms, which still remains a major open problem.

Envy-Freeness. Envy-freeness has traditionally been considered a very important fairness criterion in Economics and Political Science in settings without money and with infinitely divisible goods [5, 34]. While generally in settings with indivisible goods, envy-free allocations do not always exist, if we allow payments, in the standard quasilinear utility setting, envy-free outcomes do exist. Money is used to compensate envious players. Formally, a mechanism is envy-free for the scheduling setting, if for every player $i \in [n]$, and for every other player $h \neq i$,

$$\sum_{j=1}^{m} a_{ij}t_{ij} - p^i \leq \sum_{j=1}^{m} a_{hj}t_{ij} - p^h,$$

where p^i, and p^h are the respective payments for the players.

Haake et al. [23] characterized the class of allocations that can be implemented in an envy-free manner, in terms of a property that is called local efficiency in [30]. This requires that the allocation must maximize the social welfare over allocations permuting the same bundles, and is necessary and sufficient for envy-free implementations. For our setting the definition is the following.

[2] We refer the reader to Chapters 9 and 12 of part II of [32] for basic definitions and discussion about valuation domains.

[3] The valuation of a player i in such settings is a function of the bundle of items (or set of tasks) that i gets, and not of the other players' bundles.

[4] For settings that allow partial allocations, there exist positive results that escape those characterizations [4, 18].

Definition 3 (Local Efficiency)**.** We say that a mechanism is *locally efficient* if the mechanism selects an allocation a, such that for all $t = (t_1, \ldots, t_n)$, and all permutations π of $[n]$, it satisfies

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} \cdot t_{ij} \leq \sum_{i=1}^{n}\sum_{j=1}^{m} a_{\pi(i)j} \cdot t_{ij}.$$

There have been many papers that considered envy-free pricing for revenue maximization problems [22, 8, 6, 3], while hardness results have been shown in [17]. Kempe et al [24] studied the case where the bidders have budget constraints. Mu'alem [30], and later Cohen et al. [14] considered bounding the performance of (non-truthful) envy-free mechanisms for makespan minimization.

Our contribution. We study envy-free and truthful mechanisms for domains with additive valuations, like scheduling. It is known [16] that this class is non-empty, as VCG with Clarke payments satisfy both conditions. Cohen et al. [15] have characterized this class in terms of a Rochet-like cycle monotonicity. In [16] the same authors studied a variation where each agent has a capacity that determines the maximum number of items that she can be assigned. They focus on VCG mechanisms, and they seek for payments that are both truthful and envy-free. Very recently, Fleischer and Wang [20] showed that for the case of *two related* machines, the only mechanism that is truthful, envy-free, scalable, anonymous, and individually rational is the VCG. Our domain is multi-dimensional *(unrelated machines)*, our results hold for many players (for two items), and we require only envy-freeness on top of truthfulness.

- We investigate the allocation functions that are both weakly monotone (truthful) and locally efficient, in the case of only two tasks, but many players. We are interested in a global, Roberts-like characterization. For the sake of more generality and simple exposition, in the technical part we allow that the t_{ij} take arbitrary real values. We show that if equal bids for the same task are excluded, then the only allocation functions in this class are affine minimizers with all $\lambda_i = 1$, and further strong restrictions on the parameters γ_a (see Theorem 1). We complete the theorem by showing a simple non affine-minimizer mechanism with singularities for three players, if equal bids of different players for the same task are allowed in the input. The characterization for general (unbounded) additive domain appears in Sections 2 and 3.
- Surprisingly, we found that our proof methods and results carry over to the scheduling domain (i.e., when all t_{ij} are positive), while they do *not* carry over to additive combinatorial auctions with two items (equivalent to our model with every t_{ij} negative)! This fact is especially interesting, given that so far the two problems have been treated as "almost" equivalent. For combinatorial auctions, we present a new non affine-minimizer mechanism for three or more players, that is continuous, truthful and envy-free. The characterization for scheduling and the implications for auctions are treated in Sections 4 and 5, respectively.
- In Section 6 we consider anonymous allocations, and show that (for any number of tasks) they all have a certain technical property. Subsequently we prove that, for $m = 2$ they must be locally efficient, which implies a characterization of anonymous mechanisms for this case.
- Since the affine minimizers of the characterization theorem are both monotone, and locally efficient, they admit a truthful payment scheme, and a (possibly different) envy-free payment scheme. Such allocation rules were termed *EF* $\cup$ *IC-implementable* mechanisms in [15]. We provide a *common* payment function, i.e., a single mechanism that is both truthful and envy-free. This shows that the obtained affine minimizers are in fact the *EF* $\cap$ *IC-implementable* allocation functions. The results about payments are presented in Section 7.

It should be emphasized that this is a genuinely multi-parameter setting. To the best of our knowledge, this is the first time that a global characterization has been proven for a scheduling-type multi-dimensional domain for more than two players. Even for the simple case of three players and two tasks, no global characterization of incentive-compatible (non-envy-free) mechanisms is known, which is considered a very important open problem. Our primary goal has been to purify the general problem with the envy-freeness constraints, so that a new, structural approach to characterization becomes feasible.

Open problems. The most important question here is, whether the non-envy-free problem, or other problem variants can be tackled by generalizing our methods. Similar results for two tasks in the non-envy-free case could possibly serve as cornerstones for the general many-tasks problem [33], as has been the case in the two-player setting [10].

Taking an opposite view, we ask the following: The counterexample of Section 5 turns out to be of similar flavor as the non affine-maximizer auction of Lavi et al. ([27] Example 4.). Note that this kind of example exists despite the envy-freeness restriction, whereas no counterexample exists for scheduling. Are there nontrivial[5] counterexamples for scheduling, (or for the unbounded domain) in the truthful, *non envy-free* case? Is the orientation of the domain crucial?

Notation and basic geometry of truthfulness. The allocation of tasks to player i is denoted by a_i, and can take the values $a_i \in \{11, 10, 01, 00\}$; the allocation a to all the players is the vector $a = (a_1, a_2, \dots, a_n)$. Further, we denote by a^{ij} the allocation giving task 1 to player i and task 2 to player j.

In a truthful mechanism, the payment of player i depends on the bid matrix t_{-i} of the *other* players, and on the allocation a_i of player i. Let $p^i_{a_i}(t_{-i})$ denote this payment. We introduce the following functions:

Notation.

$$f_i(t_{-i}) = p^i_{11}(t_{-i}) - p^i_{01}(t_{-i})$$
$$f'_i(t_{-i}) = p^i_{10}(t_{-i}) - p^i_{00}(t_{-i})$$
$$g_i(t_{-i}) = p^i_{11}(t_{-i}) - p^i_{10}(t_{-i})$$
$$g'_i(t_{-i}) = p^i_{01}(t_{-i}) - p^i_{00}(t_{-i})$$

Most of the time we will apply the short notation f_i, f'_i, g_i, g'_i, and for player $i = 1$ we omit the subscript, using only f, f', g, g', for the respective values. It is well known (see, e.g.,[10]) that in any truthful mechanism, for fixed t_{-i} the allocation of player i as a function of (t_{i1}, t_{i2}) has a geometrical representation of one of three possible shapes – see Figure 2 –, where the two vertical boundaries are on the lines $t_{i1} = f_i$ and $t_{i1} = f'_i$, and the horizontal ones are on the lines $t_{i2} = g_i$ and $t_{i2} = g'_i$. Furthermore, $f'_i - f_i = g'_i - g_i$ holds. We call the 45° boundary 10/01 or 11\00 the *flipping boundary*, since there the allocation of both tasks gets flipped (the flipping boundary may happen to be a single point). Our proofs are based on this type of representation.

Notation. Let t^m be the point (in general not a single bid) with coordinates $t^m_1 = \min_{k\neq 1,2} t_{k1}$ and $t^m_2 = \min_{k\neq 1,2} t_{k2}$. Furthermore, let $M = \min_{i\neq 1}\{t_{i1} + t_{i2}\}$.

2 Constraints due to envy-freeness

We start by investigating the (geometric) restrictions that envy-freeness imposes on the possible allocations. Without loss of generality, we consider the allocation figure of player 1. In the next propositions we deal with the cases, when in t_{-1} a single player (assume wlog. player 2) bids minimum, respectively when different players (say players 2, and 3) bid minimum for the two tasks.

Proposition 1. *Assume that $t_{22} < t_{i2}$, and $t_{21} < t_{i1}$ for every player $i \neq 1, 2$. The following restrictions are implied by local efficiency (see Figure 1 (a)). If the allocation of player 1 is*

(a) (11) *then* $t_{11} + t_{12} \leq t_{21} + t_{22}$;

(b) (10) *then* $t_{11} + t_{22} \leq t_{12} + t_{21}$, *and* $t_{11} \leq t^m_1$;

(c) (01) *then* $t_{11} + t_{22} \geq t_{12} + t_{21}$, *and* $t_{12} \leq t^m_2$;

(d) (00) *then* $t_{11} + t_{12} \geq t_{21} + t_{22}$.

Proof. The proof consists of direct applications of the local efficiency property. By this, (a) and the first part of (b) and (c) are straightforward. Assume w.l.o.g. that $t^m_1 = t_{31}$ and $t^m_2 = t_{32}$. Note that in case (b) the allocation can only be a^{12}, since $t_{22} < t_{i2}$ for $i \neq 1, 2$. Comparing this with the possible allocation a^{32}, we obtain $t_{11} \leq t_{31} = t^m_1$. The proof of (c) is analogous. Finally, it is easy to see that in the area $t_{11} + t_{12} \leq t_{21} + t_{22}$ no allocation giving 00 to player 1 can be locally efficient, which yields (d). □

[5]The known non- affine-minimizers (over some subdomain) have additive payments, and are 'practically' task-independent (over the subdomain) [10, 19].

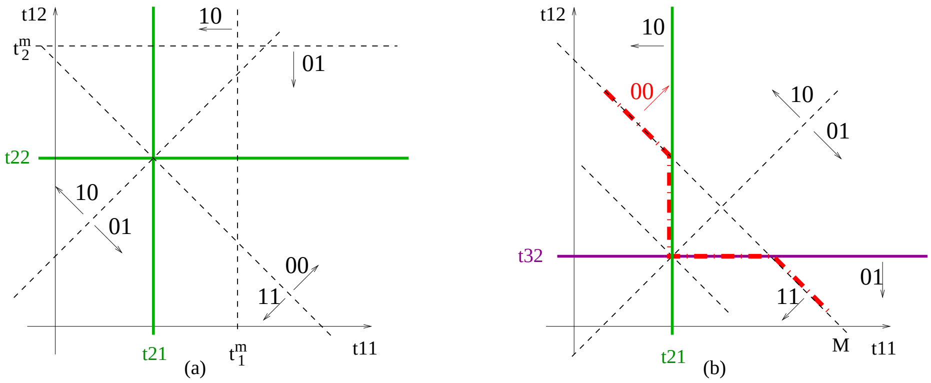

Figure 1: Envy-freeness constraints on the allocations, in case of minimum bids by a single agent (a) and two different agents (b)

Proposition 2. *Assume that $t_{21} \le t_{i1}$, and $t_{32} \le t_{i2}$ for every player $i \ne 1$. The following restrictions are implied by local efficiency (see Figure 1 (b)). If the allocation of player 1 is*

(a) (11) *then* $t_{11} + t_{12} \le M$;

(b) (10) *then* $t_{11} + t_{32} \le t_{12} + t_{21}$, *and* $t_{11} \le t_{21}$;

(c) (01) *then* $t_{11} + t_{32} \ge t_{12} + t_{21}$, *and* $t_{12} \le t_{32}$;

(d) (00) *then* $t_{11} + t_{12} \ge M$, or ($t_{11} \ge t_{21}$ *and* $t_{12} \ge t_{32}$).

Proof. (a) is immediate by local efficiency. For (b) we observe that in this case the sum of winning bids is $t_{11} + t_{32}$; comparing it to the allocations a^{21} and a^{23} yields the conditions in (b), and (c) is analogous. Finally, (d) follows by assuming that the 00 allocation is of type a^{ii}, and a^{ij} ($i \ne j$), respectively. □

The geometric implications for envy-free allocations are summarized by Corollary 1 below. They admit allocations of types shown in Figures 2 and 3.

Corollary 1. *For the allocation of player 1 in a truthful and envy-free mechanism the following hold:*

If $t_{21} < t_1^m$, and $t_{22} < t_2^m$, then the point t_2 is on the flipping boundary, furthermore $f' \le t_1^m$ and $g' \le t_2^m$.

If in t_{-1} players 2 and 3 bid minimum for tasks 1 and 2 respectively, then either $f' = t_{21}$ and $g' = t_{32}$, OR $f' \le t_{21}$ and $g' \le t_{32}$, and the flipping boundary $(11\backslash 00)$ is on the line $t_{11} + t_{12} = M$.

3 Characterization of envy-free truthful mechanisms

The characterization has two major steps. For taking the first step, we look at the case when in t_{-1} a *single* player has minimum bids for both jobs. For the second step, we examine the situation when *different* players bid minimum for the two jobs.

Focusing on the case of minimum bids by a single player, we prove that the distances $f' - t_{21}$, $t_{21} - f$, $g' - t_{22}$, and $t_{22} - g$ are independent of t_2, that is, by moving t_2 the allocation figure moves along with t_2 while keeping its shape (cf. Figure 2).

By looking at the case of minimum bids by different players (Figure 3), it becomes clear that many of these constant distances must be equal, further implying that they are even independent of all other bids (e.g., $f' - t_{21}$, is independent not only of t_2 but even of t_{-12}, the input of all players other than 1 and 2). Therewith the allocation rule turns out to be identical to that of an affine minimizer, given that all payment functions are continuous. If arbitrary functions are allowed, then it is an affine minimizer over inputs with pairwise different bids, with possible singularities when some bids are equal.

3.1 The same player bids minimum for each job

Assuming minimum bids by a single player (say, player 2) in t_{-1}, we prove that the distances $f' - t_{21}$, $t_{21} - f$, $g' - t_{22}$, and $t_{22} - g$ are independent of t_2. To be precise, this holds as long as $f' < t_1^m$, $g' < t_2^m$ and $t_{21} + t_{22}$ is minimum, i.e., $t_{21} + t_{22} = M$. In fact, these conditions delineate a slightly extended or

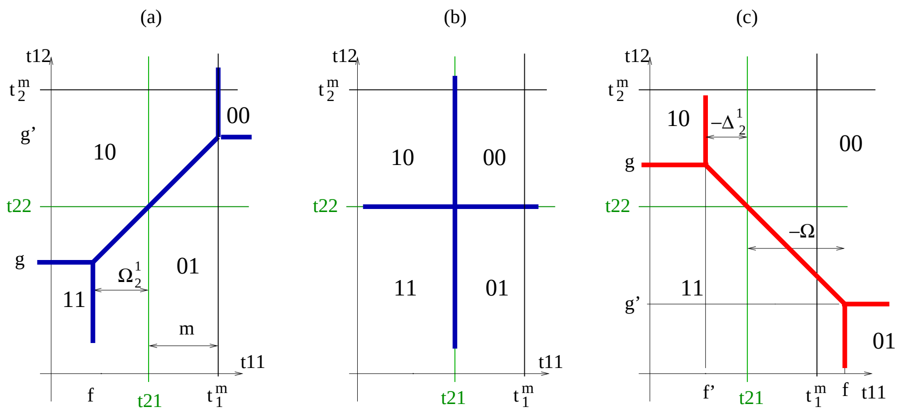

Figure 2: Possible forms of allocations when a single player bids minimum for both tasks.

narrowed domain for t_2 (as compared to the domain where he bids minimum), enforcing a somewhat technical formulation of the two lemmas.

We start with the observation that whenever player 1 is sure to exchange the first task with player 2, the functions f and f' are non-decreasing in t_{21} *independently* of t_{22}.[6] Note that the conditions of the next Lemma guarantee that the task gets exchanged with player 2, and not with some other player. Below we omit the fixed constant t_{-12} from the argument of these functions.

Lemma 1. *(a) For any t_2, t_2' s.t. $t_{21} < t_{21}' < t_1^m$, it holds that $f(t_2) \le f(t_2')$.*

(b) Let t_2 be so that $t_{22} < t_{i2}$ and $t_{21} + t_{22} < t_{i1} + t_{i2}$ for $i \ge 3$, and let the same hold for t_2'. If $t_{21} < t_{21}'$, and $g'(t_2') < t_2^m$, then $f'(t_2) \le f'(t_2')$.

Analogous statements hold for g and g'.

Proof. (a) Assume for contradiction that $t_{21} < t_{21}' < t_1^m$, and $f(t_2) > f(t_2')$. We fix a t_1 such that $f(t_2') < t_{11} < f(t_2)$, and $t_{12} < \min\{g(t_2), g'(t_2), g(t_2'), g'(t_2')\}$. For input (t_1, t_2) the allocation is a^{11} (by definition of $f(t_2)$); whereas for (t_1, t_2') it is a^{21} (by definition of $f(t_2')$ and since t_{21}' is minimum). However, for the fixed t_1, t_{-12} this contradicts truthfulness for inputs t_2 (getting 00) and t_2' (getting 10) (see Figure 2).
(b) The proof of monotonicity of f' is analogous. By local efficiency, the f' separates the allocations a^{12} and a^{22}. More precisely, an appropriate t_1 imposing these allocations for t_2 and t_2' can be found *unless* either $f'(t_2') = t_1^m$, or $g'(t_2') = t_2^m$. In the former case $f'(t_2')$ is maximum possible by Corollary 1, so $f'(t_2') \ge f'(t_2)$; the latter case violates the conditions of the lemma (and monotonicity does not hold). ☐

Lemma 2 completes the first main step of the characterization.

Lemma 2. *In every truthful envy-free mechanism, for fixed t_{-12} there exist constants $\Delta = \Delta(t_{-12})$, and $\Omega = \Omega(t_{-12})$ so that for every t_2 such that $t_{21} + t_{22} = M$,*

(a) $f(t_2) = t_{21} - \Omega$ if $t_{21} < t_1^m$ and $t_{22} \le \max\{t_2^m, t_2^m - \Omega\}$, and

(b) $f'(t_2) = \min\{t_{21} + \Delta, t_1^m\}$ whenever $t_{22} \le \min\{t_2^m, t_2^m - \Delta\}$.

Furthermore, if Δ is positive (negative) then Ω is non-negative (non-positive).

Analogous statements hold for g and g'. In particular, given that t_2 is on the flipping boundary, we obtain that if Ω and Δ are non-negative, then $g = t_{22} - \Omega$, and $g' = \min\{t_{22} + \Delta, t_2^m\}$; if they are non-positive, then $g = t_{22} - \Delta$, and $g' = \min\{t_{22} + \Omega, t_2^m\} = t_{22} + \Omega$.

An intuitive proof of the lemma is the following. By Lemma 1, $f(t_{21}, t_{22})$ is a monotone function of t_{21}, and therefore it is continuous in almost all t_{21} (say, for fixed t_{22}). Moreover, since f is monotone in t_{21} *regardless* of t_{22}, it is necessarily independent of t_{22} (that is, $f(t_{21}, t_{22}) = f(t_{21}, t_{22}')$), whenever it is continuous in t_{21}. Assume e.g., that $f(t_{21}, t_{22}) < t_{21}$. Then the fact that t_2 is on the flipping boundary (cf. Corollary 1), and $f(t_{21}, t_{22})$ is independent of t_{22} implies that $t_{22} - g(t_{21}, t_{22}) = t_{21} - f(t_{21}, t_{22})$ is a constant for fixed t_{21}. We name this constant Ω, and obtain that $g(t_{21}, t_{22}) = t_{22} - \Omega$ must hold for *all* t_{21}, since g is monotone in t_{22} regardless of t_{21}.

[6] In a more special form, the same was observed in [10].

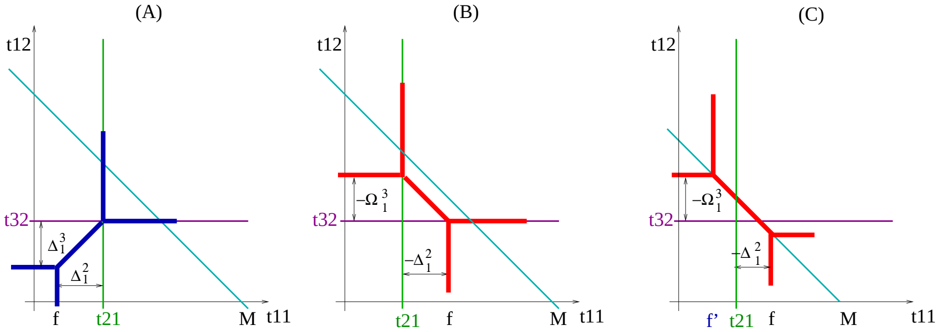

Figure 3: Possible forms of allocations when different players bid minimum for both tasks.

Proof. (a) Observe that by Corollary 1, in the given domain t_2 is on the flipping boundary (using also $t_{21}+t_{22}=M$).Let an arbitrary $t^*_{22} \le t^m_2$ be fixed, and t^*_{21} be a point where the monotone $f(t_{21}, t^*_{22})$ is continuous. We claim that $f(t^*_{21}, t_{22})$ is independent of t_{22}, that is, for every $t_{22} < t^m_2$ the equality $f(t^*_{21}, t_{22}) = f(t^*_{21}, t^*_{22})$ holds. Assume for contradiction that e.g., $f(t^*_{21}, t_{22}) < f(t^*_{21}, t^*_{22})$. Then by the continuity of f in (t^*_{21}, t^*_{22}), there is an ϵ, such that $f(t^*_{21}, t_{22}) < f(t^*_{21} - \epsilon, t^*_{22})$, contradicting Lemma 1. The proof is analogous if we assume $f(t^*_{21}, t_{22}) > f(t^*_{21}, t^*_{22})$.

Now define Ω as $\Omega = t^*_{21} - f(t^*_{21}, t^*_{22})$. Assume first, that $\Omega > 0$. Since t_2 is on the flipping boundary, in this case f uniquely determines g, in particular $t_{22} - g(t^*_{21}, t_{22}) = t^*_{21} - f(t^*_{21}, t_{22}) = t^*_{21} - f(t^*_{21}, t^*_{22}) = \Omega$, that is, for all t_{22} we have $g(t^*_{21}, t_{22}) = t_{22} - \Omega$. Observe that for the fixed t^*_{21} the function $g(t^*_{21}, t_{22})$ is continuous in *every* point $t_{22} < t^m_2$ so that for arbitrary fixed t_{22}, by exchanging the roles of f and g, and of t^*_{22} and t^*_{21}, (using Lemma 1 for g) we obtain $f(t_{21}, t_{22}) = t_{21} - \Omega$ for all $t_{21} < t^m_1$ and $t_{22} < t^m_2$. Second, if $\Omega \le 0$, we replace g by g' and use an analogous argument. Note that the argument holds as long as $g' < t^m_2$ which, in turn, holds if $t_{22} < t^m_2 - \Omega$.

(b) In order to show (b), we need some technical preparation. We search for a point of the form $t_2 = (t^m_1 - y, t^m_2 - y)$ for which $f'(t_2) < t^m_1$ (if such a point does not exist, that corresponds to the degenerate case of $\Delta = \infty$). Next, let $t^*_{22} = t^m_2 - y - \epsilon$. Now for all points (t_{21}, t^*_{22}) such that $t^m_1 - y - \epsilon < t_{21} < t^m_1 - y$ it holds that $f' < t^m_1$, by monotonicity of f' (Lemma 1); and also $g' < t^m_2$ by trivial geometry. We fix a point t^*_{21} in this interval, where the (nondecreasing) $f'(t_{21}, t^*_{22})$ is continuous. From here on, the proof is basically the same as above: We show that for arbitrary $t_{22} < t^m_2$ the equality $f'(t^*_{21}, t_{22}) = f'(t^*_{21}, t^*_{22})$ holds, unless $g'(t^*_{21}, t_{22}) = t^m_2$. Then, Δ is defined as $f'(t^*_{21}, t^*_{22}) - t^*_{21}$; finally, the monotonicity of g and g' is exploited in the cases $\Delta < 0$ and $\Delta \ge 0$, respectively. ($g' = t^m_2$ occurs exactly when $t_{22} \ge t^m_2 - \Delta$, otherwise the monotonicity of f' or g' is violated.)

For (b) we exploited the fact that by Corollary 1, in the given domain either t_2 is on the flipping boundary, or $f' = t^m_1$ holds. □

Remark 1. The above proof is a variant of the characterization proof of [10] for two players. Here, the fact that t_2 is on the flipping boundary allows a simplification. Curiously, even the non-envy-free case admits a comparably simple argument showing independency and linarity of f and g (given that $f' \neq f$ in some continuous point).

Notation. Let $\Delta^1_2 = \Delta^1_2(t_{-12})$, and $\Omega^1_2 = \Omega^1_2(t_{-12})$ denote the constants obtained in Lemma 2. For arbitrary two players $i \neq j$ we define $\Delta^i_j(t_{-ij})$, and $\Omega^i_j(t_{-ij})$ analogously. Note that Δ^i_j and Ω^i_j appear in the allocation figure of player i, when player j has minimum bids in t_{-i}.

Observation 1. *For any $i \neq j$, for fixed t_{-ij} we have $\Delta^i_j = \Omega^j_i$.*

Proof. For example, let t_{-12} be fixed, and $\Delta = \Delta^1_2 > 0$. We show that $\Delta = \Omega^2_1$ (see Figure 4 (b)). Consider a t_2 s.t. $f'(t_2) = t_{21} + \Delta < t^m_1$ and $g'(t_2) = t_{22} + \Delta < t^m_2$ (if such a t_2 does not exist, then $\Delta = \Omega^2_1 = \infty$ can be shown). Now let $t_{11} = t_{21} + \Delta + \delta < t^m_1$, and $t_{12} = t_{22} + \Delta - \epsilon$. Since t_1 has minimum coordinates in t_{-2}, Ω^2_1 appears in the allocation figure of player 2. By the position of t_1, the allocation is a^{21} for t_1, t_2. Therefore, in the figure of player 2, $\Omega^2_1 \ge t_{12} - t_{22} = \Delta - \epsilon$. By $\epsilon \to 0$ we get $\Omega^2_1 \ge \Delta$. Similarly, by setting $t'_{12} = t_{22} + \Delta + \epsilon$, we obtain $\Omega^2_1 \le \Delta$. □

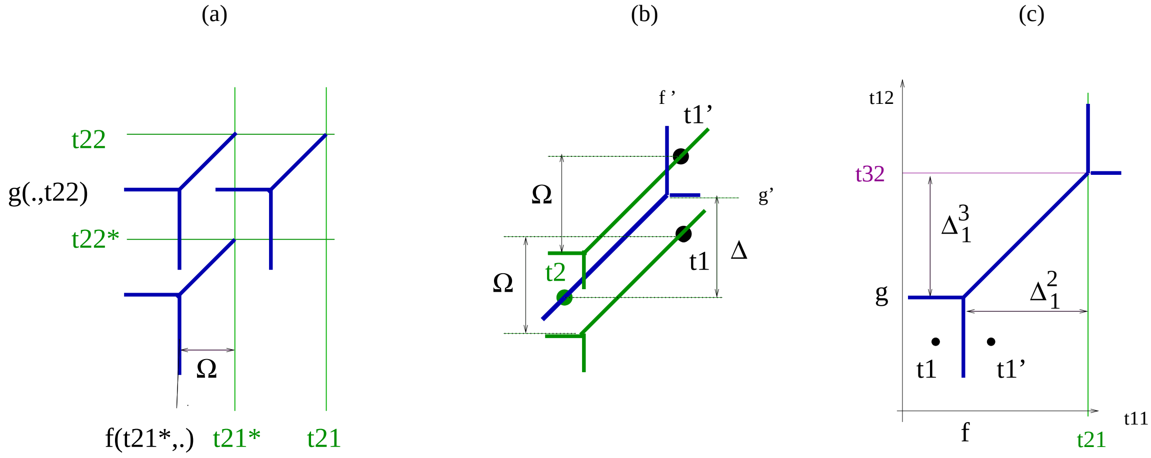

Figure 4: Illustration to the proofs of Lemma 2, of Observation 1, and of Lemma 3.

3.2 Different players bid minimum for the two jobs

Here we consider the allocation to player 1 when in t_{-1} different players bid minimum for tasks 1 and 2, and conclude that the $\Delta^i_j(t_{-ij}) = \Omega^j_i(t_{-ij})$ values are independent of (t_{-ij}). We complete the characterization by showing that even many of these constants are equal, and either all Δ are non-negative, or all are non-positive.

Lemma 3. *Let t_{-1} be fixed so that $t_{21} < t_{i1}$ for all $i \neq 1, 2$, and $t_{32} < t_{i2}$ for all $i \neq 1, 3$. Then for the allocation of player 1 $f = t_{21} - \Delta^2_1(t_{-12})$ (see Figure 3). Furthermore, $g = t_{32} - \Delta^3_1(t_{-13})$ if $g < t_{32}$, and $g = t_{32} - \Omega^3_1(t_{-13})$ if $g > t_{32}$, and at least one of these two holds if $g = t_{32}$.*

Proof. Let $t_1 = (f - \epsilon, t_{12})$, and $t'_1 = (f + \epsilon, t_{12})$ where $t_{12} < \min\{g, g'\}$ (see Figure 4 (c)). Then, by definition of f, and because t_{21} is strict minimum, the allocation is a^{11} for input (t_1, t_{-1}), and a^{21} for input (t'_1, t_{-1}). Now we change our point of view, and look at the allocation figure of player 2. Since with input t_1 player 2 receives 00, and for t'_1 he receives 10, we have

$$f'_2(t_1) \leq t_{21} \leq f'_2(t'_1).$$

Lemma 2 adapted for player 2 states that $f'_2(t_1) = \min\{t_{11} + \Delta^2_1, t^m_1\}$ whenever $t_{12} \leq \min\{t_{32}, t_{32} - \Delta^2_1\}$, and $t_{11} + t_{12}$ is minimum sum of bids in t_{-2}. Assuming that the conditions of the lemma hold for both t_1 and t'_1, we have $f'_2(t_1) = \min\{t_{11} + \Delta^2_1, t^m_1\}$, and $f'_2(t'_1) = \min\{t'_{11} + \Delta^2_1, t^m_1\}$. Now notice that $t^m_1 = f'_2(t_1) \leq t_{21}$ would contradict $t^m_1 > t_{21}$ (here we exploit that t_{21} is *strict* minimum in t_{-1}). Therefore,

$$f'_2(t_1) = t_{11} + \Delta^2_1,$$

and

$$f'_2(t'_1) \leq t'_{11} + \Delta^2_1.$$

Putting all inequalities together, we obtain $t_{11} + \Delta^2_1 \leq t_{21} \leq t'_{11} + \Delta^2_1$. This implies the lemma, since $t_{11}, t'_{11} \to f$ if $\epsilon \to 0$.

It remains to prove the conditions of Lemma 2 for t_1 and t'_1. The only condition that needs verification is $t_{12} \leq t_{32} - \Delta^2_1$ if Δ^2_1 is positive (note that $t_{12} = t'_{12}$). With input t_1 player 2 receives 00. We choose t_1 so that $t_{32} - t_{12} > t_{21} - t_{11}$. In this case the allocation 00 to player 2 is possible only if $t_{21} - t_{11} = \Delta^2_1$ (cf. Figure 2 (a)). Thus, $t_{32} - t_{12} > \Delta^2_1$ and we are done.

The second statement of the lemma follows by a symmetric argument (the difference in formulation is due to the task-specific definition of Δ and Ω). □

Lemma 4. *Let $t_{-12} = (t_3, t_4, \ldots, t_n)$, and $t'_{-12} = (t'_3, t_4, \ldots, t_n)$ be such that $\max\{t_{32}, t'_{32}\} < t_{i2}$ (resp. $\max\{t_{31}, t'_{31}\} < t_{i1}$) for $i \neq 1, 3$. Then $\Delta^2_1(t_{-12}) = \Delta^2_1(t'_{-12})$.*

Proof. Assume first, that $\Delta^2_1(t_{-12}) > 0$. Fix an arbitrary t_2 such that $t_{22} > \max\{t_{32}, t'_{32}\}$, and t_{21} is smaller than all first bids (see Figure 5 (a)). Now t_{32} is minimum among second bids, moreover $\Delta^2_1(t_{-12}) > 0$, and by the previous lemma $f = t_{21} - \Delta^2_1(t_{-12})$; therefore the allocation can only have the form as in Figure 3 (A). Consequently, $\Delta^2_1(t_{-12}) = t_{21} - f = t_{32} - g = \Delta^3_1(t_{-13})$. Now, since $\Delta^3_1(t_{-13}) > 0$, by exchanging the roles of players 2 and 3, and tasks 1 and 2, we obtain that also $\Delta^3_1(t_{-13}) = \Delta^2_1(t'_{-12})$.

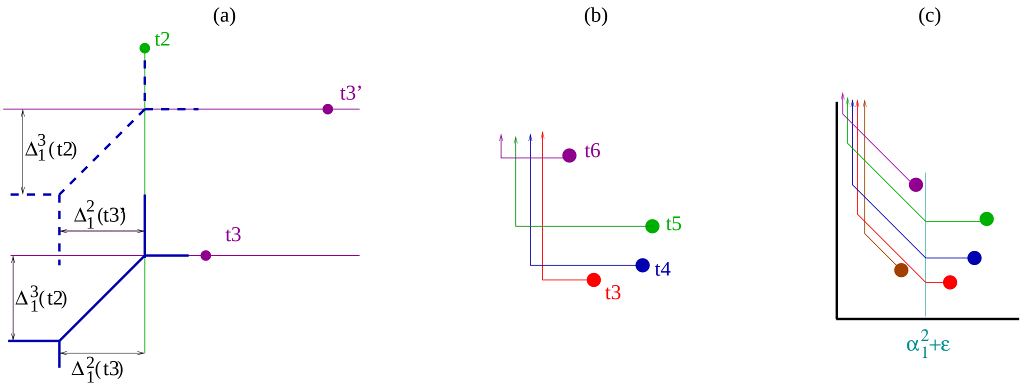

Figure 5: Illustration to the proofs of Lemma 4, Lemma 5, and Theorem 2.

Second, let $\Delta_1^2(t_{-12}) < 0$. Note that in this case, in order to have $f - t_{21} = g - t_{32}$ we have to make sure that the allocation is of type (B), and not of type (C) in Figure 3. The only bid in t_{-1} that we are free to choose is t_2. We choose a t_2 such that t_{22} is large and t_{21} is minimum, moreover even $t_{21} + \max\{t_{32}, t'_{32}\} - \Delta_1^2(t_{-12}) < M$, holds. (This becomes problematic if we allow only positive or only negative t_{ij}.) Then the allocation must be of type (B), which implies $-\Delta_1^2(t_{-12}) = f - t_{21} = g - t_{32} = -\Omega_1^3(t_{-13})$. Since $\Omega_1^3(t_{-13})$ is negative, and t_{21} is small enough, for input t'_3 the allocation is of type (B) again, and we obtain $-\Omega_1^3(t_{-13}) = -\Delta_1^2(t'_{-12})$.

If $\Delta_1^2(t_{-12}) = 0$, then (set t_2 as above) $f = t_{21}$ and $g = t_{32}$. The latter happens if either $\Delta_1^3(t_{-13}) = 0$ and $\Omega_1^3(t_{-13}) \geq 0$ or $\Omega_1^3(t_{-13}) = 0$ and $\Delta_1^3(t_{-13}) \leq 0$. For input t'_3 both imply $g = t'_{32}$, which in turn yields $f = t_{21}$ for this input as well, and $\Delta_1^2(t'_{-12}) = 0$.

The case $\max\{t_{31}, t'_{31}\} < t_{i1}$ is symmetric. Note however that if $\Delta_1^2(t_{-12}) < 0$ then $\Omega_2^1 (= \Delta_1^2)$ can be shown to be constant, and players 1 and 2 change roles. □

Intuitively, the lemma implies that given the set of points $\{t_3, t_4, \ldots, t_n\}$ in the plane, we can move around the point in the lowermost (leftmost) position, the $\Delta_1^2(t_{-12})$ does not change as long as the point remains in the lowermost (leftmost) position. Next we show that an arbitrary array of points $(t_3, t_4, \ldots, t_n)$ can be transformed to another arbitrary array $(t'_3, t'_4, \ldots, t'_n)$ using only such movements. Consequently, $\Delta_1^2(t_{-12})$ is independent of t_{-12} (at least for t_{-12} where the points have pairwise different coordinates). This holds obviously for arbitrary pair of players $i \neq j$.

Lemma 5. *Let $t_{-12} = (t_3, t_4, \ldots, t_n)$ and $t'_{-12} = (t'_3, t'_4, \ldots, t'_n)$ be arbitrary such that the second (or first) coordinates of the points are pairwise different in t_{-12} and similarly in t'_{-12}. Then $\Delta_1^2(t_{-12}) = \Delta_1^2(t'_{-12})$.*

Proof. We transform the array $(t_3, t_4, \ldots, t_n)$ into the array $(t'_3, t'_4, \ldots, t'_n)$ by moving either the leftmost or the lowermost point in every step. According to Lemma 4, this will prove the lemma. Assume w.l.o.g. that $0 < t_{32} < t_{42} < \ldots < t_{n2}$ and $0 < t'_{\pi(3)2} < t'_{\pi(4)2} < \ldots < t'_{\pi(n)2}$, where π is a permutation, and that also all first coordinates are positive. Clearly, we can move the points one after the other in the positions $(-i, \infty)$ for $i = 3, 4, \ldots, n$, by first moving them horizontally, and then vertically (see Figure 5 (b)). We claim that we can then permute them to take the positions $(-\pi(i), \infty)$. Finally, we move them back to $t'_{\pi(n)}, t'_{\pi(n-1)}, \ldots$.

In order to exchange $(-i, \infty)$ and $(-(i+1), \infty)$, we first move $(-k, \infty)$ to $(0, k)$ for $k = n, n-1, \ldots, i+1$. Then the movements $(-(i+1), \infty) \to (-i+1/2, 0)$ and $((-i, \infty) \to (0, -1))$ followed by $(-i+1/2, 0) \to (0, -2)$ exchange the vertical positions of the two points, and we put all the points back to ∞ with the points in positions $(-i, \infty)$ and $(-(i+1), \infty)$ exchanged. □

As a concluding step, we investigate the question, to what extent the constants Δ_j^i determine each other. If $\Delta_j^i = 0$ for all $i \neq j$, then the allocation is obviously the VCG allocation. Assume now that there exist two players $h \neq k$, such that $\Delta_k^h \neq 0$. We have the following corollaries of Lemmas 3 and 5.

Corollary 2. *If $\Delta_k^h > 0$, then $\Delta_k^i = \Delta_k^h$ for every player i.*

Proof. Fix a t_{-k} so that t_{h1} is strict minimum among first, and t_{i2} is strict minimum among second coordinates. Then by Lemmas 3 and 5, $0 < \Delta_k^h = \Delta_k^h(t_{-hk}) = \Delta_k^i(t_{-ik}) = \Delta_k^i$. □

Corollary 3. *If $\Delta_k^h < 0$, then for each pair i, j of different players $\Delta_j^i = \Omega_i^j \leq 0$. Furthermore, for $n \geq 3$, all of the $\Delta_j^i = \Omega_i^j$ values are equal.*

Proof. For $n = 2$ we observe that if $\Delta_k^h < 0$, then also $\Omega_k^h = \Delta_h^k \le 0$. Let $n \ge 3$. For each $i \ne h, k$ there exists a vector t_{-k} so that by Lemma 3 $\Delta_k^h = \Delta_k^h(t_{-hk}) = \Omega_k^i(t_{-ik}) = \Omega_k^i$. For $n \ge 4$, let $j \notin \{i, h, k\}$, then $\Delta_k^h = \Omega_k^j = \Delta_k^i = \Omega_i^k = \Delta_i^h$, etc. proves that all Δ and Ω are equal. For $n = 3$ the proof of equality follows from Lemma 6. ☐

Corollary 4. *Either all Δ_k^h are non-negative, or all are non-positive.*

3.3 Main theorem

We summarize our results in terms of affine minimizers, using the notation of Definition 1. Notice that if we assume continuous payment functions, then all f and g functions are also continuous, and therefore the characterization can be extended to the whole domain. (Observe also that we require a rather weak form of continuity.)

Theorem 1. *For domains with additive valuations with two tasks (items), and any number of players, any allocation rule that admits both truthful and envy-free mechanisms is an affine minimizer over the part of the domain where no two bids for the same task (item) are equal. The affine minimizer has parameters $\lambda_i = 1$ ($i \in [1, n]$), and either*
(1) $\gamma_{a^{ii}} \ge 0$, and $\gamma_{a^{ij}} = 0$ for $i \ne j$; or
(2) $\gamma_{a^{ij}} \ge 0$, and $\gamma_{a^{ii}} = 0$. Furthermore, for $n \ge 3$ all $\gamma_{a^{ij}}$ ($i \ne j$) are equal.

Assuming that (fixing the rest of the input) the payments are continuous functions of every bid t_{ij}, the allocation is an affine minimizer over the whole domain.

Proof. Having the Δ_j^i values constant, it is straightforward to verify that restricted to inputs having pairwise different bids for each task, the allocation of the mechanism is identical to that of an affine minimizer with $\lambda_i = 1$ for all i :

If all $\Delta_i^j \ge 0$, then the allocation in Figure 2 (a) corresponds to $\Delta_i^j = \gamma_{a^{ii}} - \gamma_{a^{ji}} = \gamma_{a^{ii}} - \gamma_{a^{ij}}$. By Corollary 2 these further equal $\gamma_{a^{ii}} - \gamma_{a^{ki}} = \gamma_{a^{ii}} - \gamma_{a^{ik}}$, and so all $\gamma_{a^{ji}}$ ($i \ne j$) must be equal. Since only the relative values of the additive γ matter, we can set $\gamma_{a^{ji}} = 0$ for $i \ne j$, which yields case (1).

If all $\Delta_i^j \le 0$, that corresponds to the case $\Delta_i^j = \gamma_{a^{ii}} - \gamma_{a^{ji}} = \gamma_{a^{jj}} - \gamma_{a^{ji}}$. Thus, $\gamma_{a^{ii}} = \gamma_{a^{jj}}$ for $i \ne j$, and we can set them all to 0. Corollary 3 then implies (2).

Finally, it takes a short check that the obtained types of allocations are, in fact, locally efficient. ☐

3.4 Counterexample with singularity

Some kind of restriction of the domain to pairwise different bids, or the continuity requirement is really necessary for the theorem to hold. Here we show a mechanism with singularity (maybe the simplest one among numerous examples) that is not an affine minimizer.

Example 1. Consider the following simple allocation rule for $n \ge 3$ players. Let A be the allocation of an affine minimizer with $\gamma_{a^{ii}} = 1$ for all i, and $\gamma_{a^{ij}} = 0$ for $i \ne j$ (i.e., case $\Delta > 0$), and define the allocation rule $a()$ to be $a(t) = A(t)$ if $t_1 \ne t_2$, and $a(t)$ be the VCG allocation if $t_1 = t_2$.[7] Moreover, if t_1 and t_2 have equal, and minimum coordinates, then players 1 and 2 must both get a job. For players $i \ne 1, 2$, for fixed t_{-i} the allocation looks either like A or like VCG, and is truthful and envy-free. For player 1 (and similarly for player 2), for fixed t_{-1}, the allocation figure is that of A. We only need to perform a straightforward check – assuming different relative positions of t_2 –, that in the single point $t_1 = t_2$, the allocation of player 1 is consistent with this figure.

4 Task scheduling

Our setting models the problem of (envy-free) unrelated scheduling mechanisms, if we restrict the t_{ij} to positive values.[8] The proofs of Lemmas 1, 2 and 3 carry over to the restricted domain with straightforward modifications. Futher, the same holds for Lemmas 4 and 5, in case $\Delta \ge 0$. Namely, in these cases either the statements hold or it can be shown that the respective boundaries at f, f' etc. do not appear in – i.e., 'slip out of'– the figure, which is still uniquely determined by the remaining boundaries.

Assume $\Delta_k^h < 0$ for some h, k and some t_{-hk}. The difficulty with this is that Lemma 4, and thus also Lemma 5 do not necessarily hold if the t_{ij} coordinates are bounded. In fact, for this reason,

[7] We could have used any affine minimizer with $\gamma_{a^{ii}} < 1$ instead of VCG; however $\gamma_{a^{ii}} > 1$ would not work.

[8] For simplicity we exclude $t_{ij} = 0$, since our results hold for continuous mechanisms, or for inputs with pairwise different coordinates.

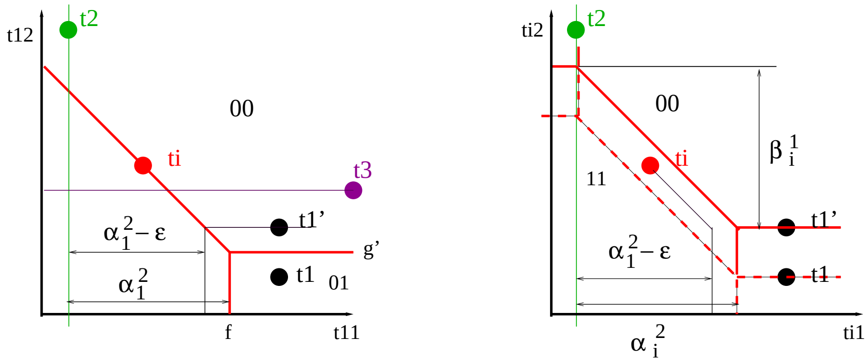

Figure 6: Illustration to Lemma 6: the allocation figures of player 1, and of player i.

the characterization result fails for $t_{ij} \in \mathbb{R}_-$ (auctions domain). Rather surprisingly, for positive t_{ij} Theorem 1 can be 'saved' thanks to Lemma 6 below. Despite that the unrelated scheduling problem and the additive combinatorial auctions problem look very similar, our results demonstrate that they do not exhibit symmetric behaviour, and are by no means equivalent problems.

Lemma 6 serves the same purpose as Lemma 3: it fixes a relation between the constants of different players who exchange tasks with player 1, whereas this time the exchange is over the flipping boundary. We introduce the notation $\alpha_k^h = -\Delta_k^h$ and $\beta_k^h = -\Omega_k^h$, and omit the t_{-hk} from the arguments. Figure 6 illustrates the proof.

Lemma 6. *Let $t_{i1} + t_{i2} < t_{k1} + t_{k2}$ for all $k \neq 1$, and t_{21} be minimum first coordinate in t_{-1}. If the allocation of player 1 is like in Figure 3 (C), and $g' > 0$ then $\alpha_1^2 = \beta_i^1$.*

Proof. In the figure of player 1, we look at the vertical boundary at f and the horizontal one at g'. We know that $f = t_{21} + \alpha_1^2$. The proof uses the points $t_1 = (f + 2\epsilon, g' - \epsilon)$ and $t_1' = (f + 2\epsilon, g' + \epsilon)$. With input t_1 the allocation is a^{21}, and with input t_1' it is a^{ii}.

Now turning to the allocation figure of player i, we observe that t_1, (resp. t_1') have minimum sum of coordiantes in t_{-i}. For t_1 player i gets 00, even though $t_{i1} + t_{i2} < t_{11} + t_{12}$. This can only happen if the allocation figure is of type Figure 3 (B) (or maybe (A)), and $\beta_i^1 = \alpha_i^2(t_1) < t_{11} - t_{21} = \alpha_1^2 + 2\epsilon$. Thus, since $t_{11} = t_{11}'$ and β_i^1 is independent of t_1, also for input t_1' we have $\beta_i^1 < t_{11}' - t_{21}$. Given that on input t_1' player i receives 11, this further implies that with input t_1' the allocation is of type (B), and $\beta_i^1 = \alpha_i^2(t_1') \geq \alpha_1^2 - \epsilon$. We obtain the claim by $\epsilon \to 0$. $\square$

Theorem 2. *Restrict the domain of bids to $t_i \in \mathbb{R}_+^2$, and consider only inputs where the t_{i1} are pairwise different, and similarly for the t_{i2}, and for the $t_{i1} + t_{i2}$. Over this domain, the truthful and envy-free mechanisms for $m = 2$ are exactly the same mechanisms as in Theorem 1.*

Proof. We provide a modified proof of Lemma 5 for the positive orthant. Assuming that $\Delta_1^2(t_{-12}) < 0$ for some t_{-12}, we show $\Delta_1^2(t_{-12}) = \Delta_1^2(t_{-12}')$. Let $t_{32} < t_{k2}$, and $t_{i1} + t_{i2} < t_{k1} + t_{k2}$ for all $k \neq 1$ (here $i = 3$ is allowed). We argue that $\alpha_1^2 = \beta_2^1$ remains constant as we transform the pointset t_{-12} into t_{-12}'. To this end, it is enough to move the points one in front of the other to positions $(\epsilon/i, \infty)$, (instead of $(-i, \infty)$ used in the proof of Lemma 5). Intuitively, after we moved the lowermost point in t_{-12} horizontally towards 0, we may have to move it along a 45° line before we can move it into ∞ vertically (see Figure 5 (c)).

Observe that α_1^2 appears in any allocation figure only if $\alpha_1^2 < t_{i1} + t_{i2}$. In this case, for small enough t_{21} the boundaries f and g' do appear in the figure of player 1, i.e., $g' > 0$. We move the points in t_{-12} one after the other into the required positions. If the allocation is of type (B), we start with the lowermost point; if it is of type (C), we start with the point of minimum sum of coordinates.

If the allocation of player 1 is of type Figure 3 (C), we move t_i up-left without changing $t_{i1}+t_{i2}$. Since β_i^1, is independent of t_i, and by Lemma 6, $\alpha_1^2 = \beta_i^1$ holds, α_1^2 (and analogously, β_1^3) does not change, and the allocation does not change either. As soon as once t_{i1} is minimum in t_{-2} and $t_{i2} > \alpha_1^2$ occurs, we fix a t_1 with large t_{11}, and $t_{12} < t_{i2} - \alpha_1^2 = t_{i2} - \beta_2^1$, and look at the figure of player 2. Here the allocation is necessarily of type Figure 3 (B) and thus $\beta_2^1 = \alpha_2^i$. The $\alpha_2^i = \beta_2^1$ remains constant, as t_{i2} is moved to ∞, and we are 'done' with point t_i .

If the figure of player 1 is of type Figure 3 (B), then we apply the previous argument twice for t_3, to first move t_3 horizontally, then (if necessary) keeping $t_{31} + t_{32}$ constant, and finally vertically.

Finally, consider the case when $\Delta_1^2(t_{-12}) > 0$ for some t_{-12}. Note that $\Delta_1^2(t_{-12})$ is relevant only if $\min_{i\neq 1,2} t_{i1} > \Delta_1^2(t_{-12})$, and $\min_{i\neq 1,2} t_{i2} > \Delta_1^2(t_{-12})$ both hold. However, two such t_{-12} vectors can be transformed in each other without changing Δ_1^2, using essentially the original argument of Lemma 5. $\square$

Notice that on $\mathbb{R}_+^2$ our mechanisms are not decisive (i.e., a single player cannot force an arbitrary outcome for himself, by bidding properly), except for the VCG mechanism.

5 Additive combinatorial auctions

In additive combinatorial auctions each player i has a positive value v_{ij} for every item j to be sold. As opposed to the cost model (scheduling), players with higher v_{ij} tend to get the item j. By using $t_{ij} = -v_{ij}$, the problem becomes equivalent to the setting used in the paper, with the t_{ij} restricted to take *negative* values. The next example shows that Theorem 1 does *not* carry over to additive combinatorial auctions. We use the notation $\alpha_k^h = -\Delta_k^h$ and $\beta_k^h = -\Omega_k^h$.

Example 2. Assume that $t_{i1} + t_{i2} \leq t_{j1} + t_{j2} \leq t_{k1} + t_{k2} \leq \ldots$ are the three smallest sums of bids over all players (break ties by player indices). Then allocate the two jobs to players i and j, according to an affine minimizer with $\alpha = -(t_{k1} + t_{k2})$ (i.e. $\gamma_{a^{ji}} = \gamma_{a^{ji}} = \alpha$, and $\gamma_{a^{ii}} = \gamma_{a^{jj}} = 0$). This mechanism is well-defined, and checking the allocation figures shows that restricted to the negative orthant, it is also truthful and envy-free.[9]

Notation. Let $T(\alpha)$ denote the closed triangular area determined by the points $(-\alpha, 0), (0, 0), (0, -\alpha)$.

The subsequent considerations intuitively tell that all counterexamples for envy-free additive auctions are variants of Example 2: changing a bid t_i to t'_i can change the α parameter to an arbitrary $\alpha' > \alpha$, only if either $t_i \in T(\alpha)$, or $t'_i \in T(\alpha)$.

The results of Section 3 hold restricted to the negative orthant (auctions domain) up to Lemma 4. Example 2 could be constructed because Lemmas 4 and 5 do not hold.[10] Instead, we can say the following (we assume pairwise different first bids, second bids resp. sums of bids in all inputs).

Lemma 7. *Let $t_{-12} = (t_3, t_4, \ldots, t_n)$ and $t'_{-12} = (t'_3, t'_4, \ldots, t'_n)$ be arbitrary sets of bids in $(\mathbb{R}_- \times \mathbb{R}_-)^n$. If $\alpha = \alpha_1^2(t_{-12}) < \alpha_1^2(t'_{-12})$, then the set of (indexed) bids of t_{-12} in the area $T(\alpha)$ is different than the set of bids from t'_{-12} in $T(\alpha)$.*

Proof. Assume by contradiction that the bidset t_{-12} restricted to $T(\alpha)$ is identical to t'_{-12} restricted to $T(\alpha)$. We claim however, that $\alpha_1^2(t_{-12})$ is invariant as we transform the set of points of t_{-12} *outside of* $T(\alpha)$ to the points of t'_{-12} outside of $T(\alpha)$, contradicting $\alpha < \alpha_1^2(t'_{-12})$. In particular, for moving a point t_i of minimum second bid $t_{i2} < -\alpha$, the proof of Lemma 4 works by choosing a $t_2 = (-K, -\epsilon)$ with K very large. Similarly, if $t_{i1} < -\alpha$ and t_{i1} is minimum, we can show that $\beta_2^1 = \alpha_1^2(t_{-12})$ is invariant by choosing a $t_1 = (-\epsilon, -K)$. First of all, we move all these points very close to $(-\alpha, 0)$, so that they have no minimum sum of coordinates.

The remaining points are outside of $T(\alpha)$, but inside the square $(-\alpha, 0) \times (-\alpha, 0)$. Such a point of minimum $t_{i1} + t_{i2}$ can be moved out of the square by keeping $t_{i1} + t_{i2}$ constant, using Lemma 6. The rest of the proof is the same as in Lemma 5. $\square$

Corollary 5. *Let $\alpha = \alpha_1^2(t_{-12})$. If t'_{-12} has the same bids (by the same players) in $T(\alpha)$, as t_{-12}, then $\alpha_1^2(t'_{-12}) = \alpha$ (see Figure 7 (a)). Also, if t_{-13} has the same bids in $T(\alpha)$, as t_{-12}, then $\beta_1^3(t_{-13}) = \alpha$.*

Corollary 6. *If a t_{-hk} exists so that all bids in t_{-hk} are outside of the triangle $T(\alpha_h^k(t_{-hk}))$, then the mechanism is an affine minimizer with $\alpha = \alpha_h^k(t_{-hk})$ restricted to the bids of each player in $(\mathbb{R}_- \times \mathbb{R}_-) \setminus T(\alpha)$.*

Notice that in Example 2 no such t_{-hk} exists; however it does exist if, e.g., we modify α to be $\alpha = \min(1, -(t_{k1} + t_{k2}))$.

[9]Note that if $t_{i1} + t_{i2} \leq t_{k1} + t_{k2}$ then for player j the area $t_{k1} + t_{k2} < t_{j1} + t_{j2}$ must be part of the 00 allocation of j. This area is bounded in the negative orthant, but not bounded if t_{ij} can take positive values. Therefore, in the scheduling or unbounded domain only degenerate affine minimizers with $\Delta = -\infty$ fulfil this requirement.

[10]The proof of Lemma 4 fails for $\Delta < 0$, because t_{22} cannot be chosen large enough.

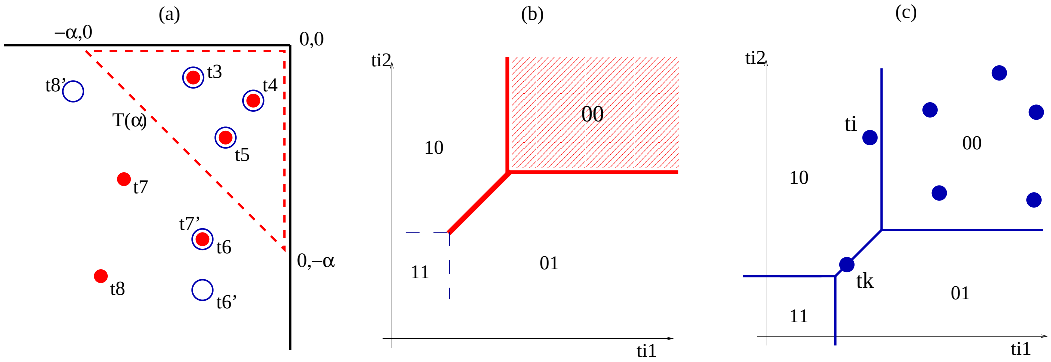

Figure 7: (a) The additive combinatorial auctions domain; bid vectors t_{-12} and t'_{-12} that induce the same $\alpha^1_2 = \alpha$ parameter. (b) The extended closure $\widetilde{R^i_{00}}$. (c) Lemma 8.

6 Anonymous mechanisms

In this section we show that in the case of two tasks, anonymous mechanisms are also locally efficient except for possible singularities. This fact, together with the above results, facilitates an easy characterization of anonymous mechanisms for $m = 2$. The first observation is an obvious consequence of anonymity.

Observation 2. *If $t_i = t_j$, in some fixed input t, then the allocation figure of player i for t_{-i} is the same as the allocation figure of player j for t_{-j}.*

Notation. For fixed t_{-i} and $c, d \in \{0,1\}^m$ let $R^i_c = R^i_c(t_{-i}) = \{t_i \in \mathbb{R}^m \,|\, a_i = c\}$, and $R^i_{cd} = \overline{R^i_c} \cap \overline{R^i_d}$ be the boundary of the regions R^i_c and R^i_d.

Definition 4. (See Figure 7 (b).) Let $\mathbf{0} \in \{0,1\}^m$ mean the zero allocation. We define the *extended closure* $\widetilde{R^i_{\mathbf{0}}}$ of $R^i_{\mathbf{0}}$ as

$$\widetilde{R^i_{\mathbf{0}}} = \overline{R^i_{\mathbf{0}}} \cup \bigcup \{R^i_{cd} \,|\, c^j = 0 \text{ or } d^j = 0 \,\forall\, j \in [1,m]\}.$$

Observation 3. *In a truthful anonymous allocation, for arbitrary number of players and tasks, for fixed i and t_{-i}, all bids of the other players are in the extended closure $\widetilde{R^i_{\mathbf{0}}}$.*

Proof. For the sake of contradiction, let $t_j \in \mathbb{R}^m \setminus \widetilde{R^i_{\mathbf{0}}}$. However, for every point $p \in R^m \setminus \widetilde{R^i_{\mathbf{0}}}$ there is some task so that i certainly gets this task if $t_i = p$. (E.g., if t_i is on the boundary of the regions R^i_{10} and R^i_{11} then he must get task 1.) Assume w.l.o.g., that player i must get task 1, if $t_i = t_j$. Then by Observation 2 player j must also get task 1 if $t_j = t_i$, a contradiction. □

6.1 Anonymous mechanisms for two tasks

Lemma 8. *For two tasks, every truthful (weakly monotone), and anonymous allocation rule is locally efficient on inputs such that $t_{i2} - t_{i1} \neq t_{j2} - t_{j1}$ for all $i \neq j$.*

Proof. Suppose we have an anonymous allocation and for given input t the allocation is a^{ii}. We claim that $t_{i1} + t_{i2} = \min_j\{t_{j1} + t_{j2}\}$, implying local efficiency for this case. Assume the contrary that $t_i \in R^i_{11}$, and still $t_{j1} + t_{j2} < t_{i1} + t_{i2}$. However, in this case, $t_j \in \widetilde{R^i_{00}}$ is impossible in any truthful allocation (see Figure 2), contradicting Observation 3.

Second, we consider the case when the allocation is a^{ij} for $i \neq j$. We show that $t_{i1} + t_{j2}$ is minimum among all $t_{h1} + t_{k2}$ $(h \neq k)$, which will complete the proof. If $t_{i1} = \min_h t_{h1}$, and $t_{j2} = \min_h t_{h2}$, then the statement is clear. Assume w.l.o.g., that $t_{k1} = \min_h t_{h1} < t_{i1}$. Then, $t_i \in R^i_{10}$ in the allocation figure of i, and by Observation 3, $t_{k1} < t_{i1}$ may occur only if $t_k \in R^i_{10,01}$ (see Figure 7 (c)). Moreover, for every other player h, the condition $t_{h2} - t_{h1} \neq t_{k2} - t_{k1}$ implies that $t_k \notin R^i_{10,01}$ but $t_h \in \overline{R^i_{00}}$. Therefore for every $h \neq i, k$ also $t_{i1} - t_{k1} \leq t_{h2} - t_{k2}$ holds. Thus, $t_{i1} + t_{k2}$ is minimum over the sums of bids of different players. If $j = k$, then we are done. If $j \neq k$, then by symmetric argument also $t_{i1} - t_{k1} = t_{j2} - t_{k2}$ must hold. However, t_k can be moved by a very small vector so that its allocation is still 00, but $t_{k1} + t_{k2} \neq t_{i1} + t_{j2}$. For this new position of t_k, no allocation is possible, a contradiction. □

Theorem 1 together with Lemma 8, and Observation 2 imply the characterization result for anonymous mechanisms for $m = 2$:

Theorem 3. *For domains with additive valuations with two tasks (items), and any number of players, the allocation of any anonymous truthful mechanism is an affine minimizer over the part of the domain where $t_{i2} - t_{i1} \neq t_{j2} - t_{j1}$, and $t_{i2} \neq t_{j2}$ (or $t_{i1} \neq t_{j1}$) for all $i \neq j$. The affine minimizer has parameters $\lambda_i = 1$ for all i, and either*
(1) there is a $\Delta > 0$ so that $\gamma_{a^{ii}} = \Delta$, and $\gamma_{a^{ij}} = 0$ for all $i \neq j$, or
(2) there is an $\alpha > 0$ so that $\gamma_{a^{ij}} = \alpha$, and $\gamma_{a^{ii}} = 0$ for all $i \neq j$.

Assuming that (fixing the rest of the input) the payments are continuous functions of every bid t_{ij}, the allocation is an affine minimizer over the whole domain.

There exist non-continuous anonymous mechanisms with singularities in inputs with $t_{i2} - t_{i1} \neq t_{j2} - t_{j1}$, that are not locally efficient and no affine minimizers.[11] For the scheduling domain, the continuous anonymous mechanisms are affine minimizers with the same restrictions on the parameters as in Theorem 3. For anonymous additive auctions Example 2 demonstrates again that anonymous continuous non-affine minimizers exist.

Remark 2. In the proof of Observation 3 we used equal bids of different players. This is *not* inconsistent either with the anonymity definition for pairwise different bids, or with the characterization result that holds for a part of the domain where bids are not equal. Still, the question remains open, whether essentially different types of anonymous truthful allocations exist if we *restrict the domain* to pairwise different bid vectors of the players. Such a mechanism for two tasks would have non-continuous payment functions, and would violate Observation 3. (A similar restriction of the input bids would not influence the envy-free characterization result.)

7 Truthful and envy-free payments

Section 3 characterized all allocation rules that are both weakly monotone, and locally efficient. Weak monotonicity implies that there exist payments that extend the allocation rule to a truthful mechanism. By local efficiency, there exist (possibly other) payment rules, so that with these the mechanism is envy-free. With the terminology introduced in [15], we characterized *EF* $\cup$ *IC-implementable* mechanisms for two tasks. In this section we provide *common* payment rules, i.e., inducing mechanisms that are both truthful and envy-free. This shows that the allocations of Theorem 1 are in fact the *EF* $\cap$ *IC-implementable* allocation functions.

Definition 5. A mechanism is *individually rational* if for every input the *profit* $p^i - a_i t_i$ of each player i is nonnegative.

Consider an allocation rule of either type (1) or type (2), as appear in Theorem 1. Recall that $\Delta_i^j = \gamma_{a^{ii}} - \gamma_{a^{ji}}$, where the γ are the additive constants of the affine minimizer. For type (2) allocations (including the VCG allocation) the normalized (Clarke-) payments will turn out to be envy-free (as has been known for VCG [16]). In contrast, for type (1) allocations with at least one $\gamma_{a^{ii}} > 0$, we prove that no individually rational payments can be both truthful and envy-free. We show examples of other (non individually rational) EF $\cap$ IC payments.

By the definition of f_i, f_i', g_i, and g_i', for any fixed player i, and fixed t_{-i} the payments that induce a truthful mechanism are exactly those of the form:

$$p^i_{01} = p^i_{00} + g_i'; \quad p^i_{10} = p^i_{00} + f_i'; \quad p^i_{11} = p^i_{00} + g_i' + f_i,$$

where $p^i_{00}(t_{-i})$ is an arbitrary real value (0 for normalized payments), and the $f_i(t_{-i})$, $f_i'(t_{-i})$, $g_i'(t_{-i})$ values are determined by the allocation function. In what follows, we search for appropriate $p^i_{00}(t_{-i})$ functions that make the mechanism also envy-free.

We start by investigating the possible $p^i_{00}(t_{-i})$, when in t_{-i} a single player j bids minimum for both tasks. Let $t_1^m = \min_{k\neq i,j} t_{k1}$ and $t_2^m = \min_{k\neq i,j} t_{k2}$. It will prove useful to search for p^i_{00} in the following form:

[11] Consider an affine minimizer with $\gamma_{a^{ii}} = 1$ for all i. If any player i has minimum bids for both tasks, and for an arbitrary j $t_{j1} = t_{i1} + 1$, and $t_{j2} = t_{i2} + 1$ hold (i.e. j is minimum player having these bids), then we give one task (1 resp. 2) to the player $k \neq i, j$ with the next smallest bid overall (t_{k1} resp. t_{k2}), and the remaining task to player i. This example is very similar to Example 1.

Notation. Let

$$D^i_j(t_j) = D^i_j[t_{-ij}](t_j) = p^i_{00} + \Delta_j(t_j),$$

where $\Delta_j(t_j) = \min\{\Delta^i_j, t^m_1 - t_{j1}, t^m_2 - t_{j2}\}$ if $\Delta^i_j > 0$, and $\Delta_j(t_j) = 0$ otherwise (i.e., for type (2) allocations, D^i and p^i_{00} are the same).

This notation yields a simple expression for the payments $p^i_{01} = t_{j2} + D^i_j(t_j)$, and $p^i_{10} = t_{j1} + D^i_j(t_j)$, also for allocations of type (1). For fixed t_{-ij}, the domain of the function $D^i_j()$ is $(-\infty, t^m_1) \times (-\infty, t^m_2)$. Our first observation largely simplifies the picture:

Lemma 9. *The function $D^i_j[t_{-ij}]()$ is independent of t_{-ij}, and of i, that is, $D^i_j(t_j) = D^k_j(t_j)$ for different i, j, k.*

Proof. Suppose first that $\Delta^i_j < \min\{t^m_1 - t_{j1}, t^m_2 - t_{j2}\}$, in which case there exists a $t_i \in (-\infty, t^m_1) \times (-\infty, t^m_2)$ so that the allocation is a^{jj}. Now, by envy-freeness, the payments of all players other than j, must be the same. It follows that $D^i_j[t_{-ij}](t_j) = D^k_j[t_{-kj}](t_j)$. Therefore, $D^i_j[t_{-ij}](t_j)$ is independent of $t_k \in t_{-ij}$ for every k (at least as long as $\Delta^i_j < \min\{t^m_1 - t_{j1}, t^m_2 - t_{j2}\}$ holds for t_{-ij}), and $D^i(t_j) = D^k(t_j)$. Suppose now that $\Delta^i_j \geq \min\{t^m_1 - t_{j1}, t^m_2 - t_{j2}\} = t^m_1 - t_{j1}$, and $t^m_1 = t_{s1}$. We can set $t_{i1} = t_{s1} - \epsilon$ and $t_{i2} = t_{s2}$. The allocation is then a^{ij}, and the envy-freeness inequalities between players i and s yield $D^i_j[t_{-ij}](t_j) = D^s_j[t_{-sj}](t_j)$, which by $D^k_j[t_{-kj}](t_j) = D^s_j[t_{-sj}](t_j)$ implies the lemma for all other players. ☐

Lemma 10. *Assume that the payments induce an envy-free mechanism. Let t_{-ij} be fixed, and t_j have minimum bids in t_{-i}, and similarly for t_i in t_{-j}. If the allocation is a^{ji}, then $t_{i2} - t_{j2} \leq D^i_j(t_j) - D^j_i(t_i) \leq t_{i1} - t_{j1}$. If it is a^{ii} then $0 \leq D^j_i(t_i) - D^i_j(t_j) \leq t_{j1} - t_{i1} + t_{j2} - t_{i2}$.*

Proof. If $t_i, t_j \in (-\infty, t^m_1) \times (-\infty, t^m_2)$ then only players i and j receive any jobs, so one of four possible allocations occurs. The statement follows directly from the envy-freeness constraints $a_i t_i - p^i \leq a_j t_i - p^j$ and $a_j t_j - p^j \leq a_i t_j - p^i$. ☐

Corollary 7. *In an envy-free mechanism for fixed t_{-ij} the $D^i_j()$ and $D^j_i()$ are identical (that is, $D^i_j(t_j) = D^j_i(t_i)$ for $t_j = t_i$), and continuous functions.*

Proof. Identity follows by applying the lemma to $t_i = t_j$, no matter if the jobs are allocated to the same or to different players. Similarly, for $t_i \to t_j$ the lemma implies $D(t_i) \to D(t_j)$ no matter what exactly the allocations are. ☐

With this we showed that for any fixed mechanism with envy-free plus truthful payments the $D^i_j[t_{-ij}]()$ functions are all restrictions of a single function $D()$, whose domain is the whole plane $\mathbb{R}^2$. Moreover, this function has the following properties.

Corollary 8. *The function $D(t_i)$ is non-increasing in t_{i1} and in t_{i2} (by some 'slope' between -1 and 0 in each point). Moreover, if all Δ^j_i are non-negative and at least one of them is strictly positive then $D((t_{i1} + \epsilon, t_{i2} + \epsilon)) = D((t_{i1}, t_{i2})) - \epsilon$. If all $\Delta^j_i < 0$, then $D((t_{i1}, t_{i2})) = D((t_{i1} - \epsilon, t_{i2} + \epsilon))$ for $\epsilon > 0$ small enough.*

Proof. The first statement is straightforward from Lemma 10 setting $t_{i2} = t_{j2}$, and $t_{i1} = t_{j1}$, respectively.

Assume that $\Delta^j_i > \epsilon > 0$ (cf. Figure 2 (a)). Then, for $t_j = (t_{i1} + \epsilon, t_{i2} + \epsilon)$ the allocation is either a^{ij} or a^{ji}, and in both cases the lemma yields $D(t_i) = D(t_j) + \epsilon$. If $\Delta^j_i < \epsilon < 0$ (cf. Figure 2 (c)), then we observe that for $t_j = (t_{i1} - \epsilon, t_{i2} + \epsilon)$ the allocation is either a^{ii} or a^{jj}, and the lemma imples $D(t_i) = D(t_j)$. ☐

For $n = 2$ players the above properties of the D function completely characterize envy-free+truthful payments, i.e. they are necessary and sufficient. Notice that – for any number of players – for type (2) allocations (including the VCG allocation) Corollary 8 admits that $D() = p^i_{00}$ is constant, and in particular that $p^i_{00} \equiv 0$. Indeed, it is straightforward to verify the following:

Theorem 4. *For type (2) allocations the truthful payments with $p^i_{00} = 0$ (Clarke-payments) are also envy-free.*

Proof. Let t_{k1} and t_{l2} be minimum first and second coordinates respectively, and let $t_{j1}+t_{j2}$ be minimum sum of coordinates in t_{-i}, where j,k,l are not necessarily different. By Theorem 1 $\gamma_{a^{li}} = \gamma_{a^{ik}}$. The Clarke-payments for player i are $p^i_{11} = \min\{t_{k1}+t_{l2}+\gamma_{a^{li}}, t_{j1}+t_{j2}\}$ $p^i_{01} = \min\{t_{l2}, t_{j1}+t_{j2}-t_{k1}-\gamma_{a^{li}}\}$ and $p^i_{10} = \min\{t_{k1}, t_{j1}+t_{j2}-t_{l2}-\gamma_{a^{li}}\}$. The envy-freeness inequalities can easily be checked for three (essentially) different pairs of players. □

Clarke-payments are also individually rational; on the other hand, as an obvious consequence of Corollary 8 we have:

Theorem 5. *For type (1) allocations with at least one $\gamma_{a^{ii}} > 0$ there exist no payments that are truthful, envy-free, and individually rational.*

Proof. By the constraint $D((t_{i1}+\epsilon, t_{i2}+\epsilon)) = D((t_{i1}, t_{i2}))-\epsilon$, the function $D(t_i)$ is negative (for example) for large enough $t_{i2} = t_{i1}$. Furthermore, for large enough coordinates of t_{-i}, (the allocation is a^{ii} and) the value $p^j_{00} = D(t_i) - \Delta_i < 0$ for every player $j \neq i$. □

From now on, we assume that the allocation is of type (1) with $\gamma_{a^{ii}} > 0$ for some player i. We find a further necessary condition for envy-free payments for $n \geq 3$. Recall that $D(t_j)$ determines p^i_{00} if j bids minimum for both tasks in t_{-i}. Consider now the setting where t_{j1} and t_{k2} are minimum among first respectively among second coordinates, furthermore both coordinates of t_i are maximum among j,k,i, and all other players have very high bids. By our characterization theorem the allocation must be a^{jk} (see Figure 3 (a)). Moreover, the payments of players j and k were determined above as $p^j_{10} = D(t_k)+t_{k1}$ and $p^k_{01} = D(t_j)+t_{j2}$. Concerning the payment p^i_{00} the envy-freeness conditions yield the bounds

$$\max\{D(t_k), D(t_j)\} \leq p^i_{00} \leq \min\{(t_{k1}-t_{j1}+D(t_k)), (t_{j2}-t_{k2}+D(t_j))\}.$$

Thus, in this case we can set $p^i_{00} = \max\{D(t_k), D(t_j)\}$, and try to find an appropriate $D()$ function (consistent with Corollary 8) that in addition yields envy-free payments with the given definitions of p^i_{00}. It turns out (by straightforward check) that the functions $D(t_j) = -t_{j1}$ (or $D(t_j) = -t_{j2}$) provide envy-free payments (whereas other functions of the form $D(t_j) = -\nu t_{j1} - (1-\nu)t_{j2}$ do not!).

Theorem 6. *Consider a type (1) allocation with truthful payments defined as: if both coordinates of t_j are minimum in t_{-i}, then let $p^i_{00} = -t_{j1} - \Delta_j(t_j)$. If t_{j1} and t_{k2} are minimum among first respectively among second coordinates in t_{-i}, then let $p^i_{00} = -t_{j1}$. The induced truthful mechanism is also envy-free.*

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